

STABILITY IN SPATIAL VOTING GAMES WITH  
RESTRICTED PREFERENCE MAXIMIZING

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**ABSTRACT:**

Spatial models of simple majority rule voting suggest that stable decisions are not likely to exist under normal circumstances. Yet this instability result stands in contrast to the stability observed in experiments. This paper examines the effect of relaxing the assumption that voting is costless by requiring a proposal to be a finite distance closer to a member's ideal point than the pending proposal before it is regarded as attractive. Using the concept of the epsilon-core the paper estimates the minimal decision costs that guarantee stable outcomes. It shows that the minimal costs are equal to the minimal finagle radius (Wuffle et al. 1989) and that the epsilon-core contains the finagle point which is close to the center of the yolk. While the analytical model establishes that the minimal costs are smaller than the yolk radius, computational simulations of majority voting by committees of size 3 to 101 suggest that this is a weak upper bound, only, as the ratio of minimal costs to the yolk radius usually is small and decreases as committee size approaches infinity.

**KEY WORDS:** spatial voting models, decision costs, epsilon-core, stability

## 1. INTRODUCTION

Majority-rule based aggregation of preferences has largely been investigated in a setting referred to as spatial model of voting in which alternative policies are viewed as points in a convex space over which a finite set of voters have single-peaked preferences. Beginning with Hotelling's (1929) theory of economic competition, Black's (1958) theory of committee decision-making, and Down's (1957) economic theory of voting, spatial models have become common in the analysis of party competition, electoral behavior, legislative and government decision-making. The most famous results, however, are discouraging with regard to democratic political processes: When alternatives are evaluated on two or more dimensions, Condorcet winners or undominated alternatives are unlikely to exist (Plott 1967; Davis et al. 1972) and cycles of pairwise voting may be constructed that include any alternative (McKelvey 1976; Schofield 1978). Based on the empirical observation that there is more stability in real world politics than theoretical models of social choice indicate (Tullock 1981), scholars have studied the conditions that guarantee or favor the existence of stable, undominated outcomes.<sup>1</sup> Following the initial focus on domain restrictions of preferences (Arrow 1951) such as single-peakedness (Black 1958), most of these studies concentrate on institutional arrangements such as super-majoritarian quotas (Greenberg 1979), agenda-setting and germaneness rules (Shepsle and Weingast 1981), bi- or multicameralism (Hammond and Miller 1987; Bräuninger 2003) that lead to some sort of structure induced equilibrium (Shepsle 1979).

This article takes a different stance. It argues that even in the absence of dense institutional environments, there are numerous factors that may create majority-rule stability in a

spatial context where it otherwise does not exist and that are related to individual's ability or willingness to rank alternatives or vote on proposals rather than restrictive institutional rules. Examples of restricted preference maximization are the bias to vote for the incumbent, well-known in studies of voting behavior, or the restricted ability of committee members "to make fine distinctions between alternatives without becoming indifferent" (McKelvey 1976: 481). While the conventional preference-maximizing assumption has often been questioned, the conjecture that voting behavior that falls short of strictly distance-based preference maximizing is likely to lessen the effects of instability, has not been studied extensively (cf. Koehler 2001: 155f.).

In this article I contribute to this literature by considering what happens when relaxing the assumption that voting is costless for any committee member. More specifically, I model committee members as individuals that do not regard a proposal as attractive unless it is a finite distance closer to their ideal point than the pending proposal. The decisive question then is whether these thresholds or decision costs will induce majority-rule stability and if so, where stable outcomes are located in the policy space. Using the concept of the  $\varepsilon$ -core the paper estimates the minimal decision costs that guarantee stable outcomes. It shows that the minimal costs are equal to the minimal finagle radius (Wuffle et al. 1989) and that the  $\varepsilon$ -core contains the finagle point which is close to the center of the yolk. While the analytical model establishes that the minimal costs are smaller than the yolk radius, computational simulations of majority voting by committees of size 3 to 101 suggest that this is a weak upper bound, only, as the ratio of minimal costs to the yolk radius usually is small and decreases as committee size approaches infinity.

## 2. PREFERENCE-INDUCED STABILITY IN SPATIAL VOTING

Indifference and alienation are two concepts that are used to analyze situations where agents abstain from voting so that stable outcomes become more likely. An agent may be indifferent when he perceives that there are only minor differences among the alternatives, whereas alienation refers to a situation where all options are outside the range of interesting, attractive alternatives (Hinich and Munger 1997: 141). Regarding voting behavior in mass elections, indifference and alienation have been incorporated in the Downsian spatial model (Riker and Ordeshook 1968; Enelow and Hinich 1984) but there are only a few studies that explore their impact on stability in situations of committee voting. Other conditions such as constraints on candidates' moves by previously publicly held positions (Samuelson 1984; see also Laver 2005) also seem to be more apt for electoral competition than committee voting.

Tovey (1991) discusses several assumptions concerning individual decision-making that each lead to a modification of the conventional spatial model and that are likely to induce stability. These include bounded rationality (individuals have limited "computational" abilities when making their choices), the presence of uncertainty as to the consequences of the proposal ("incumbency advantage"), or perceptual thresholds where individuals are not able to distinguish between proposals that are too similar. In the same vein, Feld and Grofman (1996) argue that voters may be neither willing to waste time considering alternatives that are only trivially different from one another nor to support large changes. They also provide analytical results on when restricted preference maximization under such "no-quiribbling" and incrementalism norms leads to stable outcomes under majority rule.

Most recently, Koehler (2001) has re-addressed the question of preference-induced stability in the context of committee voting. He considers thresholds or hurdles of minimal improvements that proposals must take to outweigh certain (though unspecified) decision or transaction costs: “If  $\varepsilon$  [the individual preference threshold] is large, members vote only for proposals that are substantial ‘improvements’ on the status quo or the pending proposal” (Koehler 2001: 160; see also Tovey 1991: 10). As a result, a proposal may attract an agent’s vote only if it is to a certain extent closer to her ideal point than the pending proposal. Decision-making outcomes are thus more likely to be stable because any further decision requires (transaction) cost-offsetting improvements on the side of a majority of committee members. This setting leads to the concept of the quasi- or  $\varepsilon$ -core developed by Shapley and Shubik (1966) in their study of pure exchange economies with participating traders having nonconvex preferences. In these markets the core is often vacuous, and in order to “resolve the inherent instability of this situation, we must resort to social, cultural, or institutional restraints.” (Shapley and Shubik 1966: 823). They consider  $\varepsilon$  as an organizational cost prerequisite to cooperative action, and quasi-cores as a way of taking into account the costs of coalition formation, for example. While the  $\varepsilon$ -core has become an established concept in the study of general  $n$ -person games with coalition structure (Maschler et al. 1979; Nakayama and Naitoh 1999; Kovalenkov and Wooders 2000) it has seldom been applied to spatial models of voting.

One exception is Salant and Goodstein (1990). Studying committee behavior in majority rule voting experiments, they consider threshold effects in order to explain why their experimental committees frequently failed to choose the Condorcet alternative. They estimate

the minimal improvements that players might consider “not worth the trouble” (Salant and Goldstein 1990: 302). Empirically they find that even for pretty small thresholds the predictive power of the modified spatial model is impressive. What is more, when applied to the experimental results of independent studies (Fiorina and Plott 1978; Eavey and Miller 1984), 95 percent of all committee choices are found in the selection set of undominated points. A second exception is Herzberg and Wilson (1991) who subject a spatial voting model incorporating a form of transaction costs to empirical testing using laboratory experimental data. They find that while the evidence for predictions of outcomes under the modified model are not overly compelling, the experiments “clearly point to the salience of agenda costs for the collective choice *process*” (Herzberg and Wilson 1991: 197). A third exception is Tovey (1991: 11-13) who argues that the  $\varepsilon$ -core may reflect the presence of costs of change but may also be justified by the limited computational abilities of agents in a bounded rationality framework (see also Sloss 1973: 21). He shows that for the class of centered distributions of ideal points the  $\varepsilon$ -core generically exists for *any*  $\varepsilon > 0$  as the number of ideal points goes to infinity (Tovey 1991: 11). For non-centered distributions, that is when the yolk radius is strictly larger than zero, the yolk center proves to be an undominated point if the threshold  $\varepsilon$  is strictly larger than the yolk radius (Tovey 1991: 20). The simulation study of Koehler supports these earlier experimental results and theoretical accounts indicating that “stability can be achieved at relatively low *individual* preference thresholds” and stable outcomes “are found near the [yolk] center of the preference space” (Kohler 2001: 166).

Given these theoretical results on finite committees drawn randomly out of (continuous) probability distributions and the simulation and experimental findings on small committees, the analytical questions that remain are, when does the  $\varepsilon$ -core in committees of finite size exist, where is it located and how large is it. In the following, some implications of the individual preference threshold setting for the existence of majority rule equilibrium are studied. In particular, I show that as the preference threshold increases, the first stable point that emerges is the so-called finagle point, not the yolk center.

### 3. COMMITTEE VOTING WITH DECISION COSTS

This section introduces a simple decision or transaction cost term to the conventional spatial model of voting. Based on modified agents' preferences the  $\varepsilon$ -core is defined. The  $\varepsilon$ -core is a superset of the conventional core and may exist when the latter is empty. It is argued that the yolk radius is an upper bound for the minimal decision costs that guarantee that stable outcomes exist. In this case the center of the yolk is an element of the  $\varepsilon$ -core.

Suppose that the policy space is given by a nonempty, compact and strictly convex subset of the multi-dimensional Euclidean space,  $X \subseteq \mathfrak{R}^k$ . The set of agents involved in decision-making within the committee is given by  $N = \{1, \dots, n\}$ ,  $n$  odd. Decisions are taken by simple majority. In the conventional spatial model, agents are assumed to have strictly convex, single-peaked preferences over the set of alternatives. This means that for any agent a so-called ideal point  $x_i$  exists that represents its most preferred policy and any other alternative is ranked lower relative to its – probably weighted – distance from the ideal point.

More specifically, assume that all ideal points  $x_i$  are unique and agents have symmetric Euclidean preferences so that an alternative  $x$  is strictly preferred to alternative  $y$ ,  $x P_i y$ , if and only if  $\|x_i - x\| < \|x_i - y\|$ . It is weakly preferred to  $y$ ,  $x R_i y$ , if and only if  $\|x_i - x\| \leq \|x_i - y\|$ .

In the conventional model, agents would vote for policy change if and only if the new policy is more beneficial than the status quo. The act of voting itself is assumed to be costless. Casting one's vote, however, requires the acquisition of information, evaluation of competing policies, deliberation and, at least, time. Introducing such decision or transaction costs to the spatial model means that the utility of a policy change must outweigh the costs that are incurred in the process of decision-making itself. Let these costs or voting threshold be denoted by  $\varepsilon$ . Throughout this paper, I assume that they are not agent-specific and independent from the decision to be made.<sup>2</sup> Then agent  $i$  (strictly) prefers an alternative  $x$  to alternative  $y$ ,  $x P_i^\varepsilon y$  if and only if  $\|x_i - x\| < \|x_i - y\| - \varepsilon$ , the agent weakly prefers  $x$  to  $y$ ,  $x R_i^\varepsilon y$ , if and only if  $\|x_i - x\| \leq \|x_i - y\| + \varepsilon$ . Note that for any  $\varepsilon > 0$ , the binary relation  $R_i^\varepsilon$  is complete, reflexive, and quasi-transitive, but not transitive. The following definition presents the standard terminology of the preferred set, the winset, and the core and extends it to  $\varepsilon$ -based preference relations.

**DEFINITION 1:** (i) The *preferred set* of  $x$  (regarding the binary relations  $P_i$  and  $P_i^\varepsilon$ , respectively) contains all policies that agent  $i$  strictly prefers to policy  $x$ :

$$P_i(x) = \{ y \in X : y P_i x \} \text{ and } P_i^\varepsilon(x) = \{ y \in X : y P_i^\varepsilon x \}.$$

(ii) Let  $W$  denote the set of winning coalitions, i.e. sets of agents that the voting rules empower to change the status quo. Say that some policy  $y$  is *socially preferred* to policy  $x$  if there is a winning coalition,  $S \in W$ , whose members all prefer  $y$  to  $x$ . The *winset* of  $x$  then is the set of all policies that are socially preferred to  $x$ :

$$W(x) = \{y \in X: \exists S \in W \text{ so that } y \in P_i(x) \text{ for any } i \in S\} \text{ and}$$

$$W^\varepsilon(x) = \{y \in X: \exists S \in W \text{ so that } y \in P_i^\varepsilon(x) \text{ for any } i \in S\}.$$

(iii) The *core* is the set of all alternatives that cannot be upset by any other alternative:

$$C = \{x \in X: W(x) = \emptyset\} \text{ and } C^\varepsilon = \{x \in X: W^\varepsilon(x) = \emptyset\}.$$

I shall illustrate these definitions in two dimensions. For example, Figure 1 shows a set of ideal points for three committee members, their preferred set of a pending proposal  $x$  for both conventional and  $\varepsilon$ -based preferences relations. For any ideal point  $x_i$  and pending proposal  $x$ , the  $\varepsilon$ -preferred set of  $x$ ,  $P_i^\varepsilon(x)$ , is a subset of the conventional preferred set,  $P_i(x)$ , and, accordingly,  $W^\varepsilon(x)$  is contained in the conventional winset,  $W(x)$ . For large values of  $\varepsilon$ , the  $\varepsilon$ -winset of  $x$  will be empty so that  $x$  is in the  $\varepsilon$ -core though the conventional core is empty. Denote the open ball with center  $x$  and radius  $\varepsilon$  by  $D(x, \varepsilon) = \{y \in \mathfrak{R}^k : \|y - x\| < \varepsilon\}$ . The following is then straightforward.

\*\*\* Figure 1 about here \*\*\*

**LEMMA 1:** For any  $i \in N$ ,  $x \in X$ :

- (i)  $P_i^\varepsilon(x) \subseteq P_i(x)$ ,  $W^\varepsilon(x) \subseteq W(x)$ , and  $C \subseteq C^\varepsilon$ .
- (ii)  $D(x, \varepsilon) \cap W^\varepsilon(x) = \emptyset$  for any  $\varepsilon > 0$ .
- (iii) Let  $\delta > 0$  so that  $W(x) \subseteq D(x, \delta)$ . If  $\varepsilon > \delta$  then  $x \in C^\varepsilon$ .

*Proof:* (i) is immediate from Definition 1.

(ii) For any  $y \in D(x, \varepsilon)$ , we know that  $|\|y - x_i\| - \|x - x_i\|| \leq \varepsilon$  for any agent  $i$ . Thus  $y \notin P_i^\varepsilon(x)$  and  $y \notin W^\varepsilon(x)$ . The open ball  $D(x, \varepsilon)$  is therefore an inner hole of  $W^\varepsilon(x)$ .

(iii) We have to show that there is no policy  $y$  so that  $y \in W^\varepsilon(x)$ . First, assume that  $y \notin D(x, \delta)$  which implies  $y \notin W(x)$ . Since  $W^\varepsilon(x) \subseteq W(x)$ ,  $y$  is not socially preferred to  $x$ . Second, let  $y \in D(x, \delta)$ . Then  $|\|y - x_i\| - \|x - x_i\|| \leq \delta < \varepsilon$  for any  $i$ . Thus, again,  $y \notin W^\varepsilon(x)$ .

This means that the  $\varepsilon$ -winset is contained in the conventional winset and the open ball  $D(x, \varepsilon)$  represents an inner hole of  $W^\varepsilon(x)$ . Note that in contrast to the conventional winset  $W(x)$ , the  $\varepsilon$ -winset  $W^\varepsilon(x)$  is not starlike and not connected.

Given the fundamental importance of stable decisions it is natural to ask, under which conditions the  $\varepsilon$ -core is not empty. To this end, I first introduce the concept of the yolk which is the smallest circle that intersects all median hyperplanes (Ferejohn et al. 1984; McKelvey 1986). The yolk radius can be considered as a measure for the extent to which the distribution of ideal points “deviates from one that would generate majority-rule equilibrium” (Miller et al. 1989: 405). From the yolk, lower and upper bounds of any point’s winset can be derived. Let  $c$  be the center and  $r$  be the radius of the yolk, and denote the distance between an arbitrary point  $x$  and the center of the yolk by  $d$ . Ferejohn et al. (1984)

show that upper and lower bounds of the winset  $W(x)$  are given by two circles around  $c$  with radii  $d-2r$  and  $d+2r$ , i.e.  $D(c,d-2r) \subseteq W(x) \subseteq D(c,d+2r)$ .<sup>3</sup> The next lemma states that there is a similar upper bound for the threshold-based winset:

**LEMMA 2:** For any  $x \in X$  and  $\varepsilon > 0$ :

$$W^\varepsilon(x) \subseteq D(c,d+2r-\varepsilon) \setminus D(x,\varepsilon),$$

where  $c$  is the center,  $r$  is the radius of the yolk and  $d$  is the distance of  $c$  and  $x$ .

*Proof:* (i) I first show that for any alternative in the  $\varepsilon$ -winset of  $x$ ,  $y \in W^\varepsilon(x)$ , there is an alternative in the conventional winset,  $z \in W(x)$ , so that  $\|c-y\| < \|c-z\| - \varepsilon$ . To this end let

$$z = y + \frac{\varepsilon}{\|y-c\|}(y-c)$$

and recall that as  $y \in W^\varepsilon(x)$  there is an agent set  $S$  so that  $\|x_i-y\| < \|x_i-x\| - \varepsilon$  for any  $i \in S$ .

Then, by the triangle inequality, we have that for any  $i \in S$ :

$$\|x_i - z\| = \left\| x_i - y - \frac{\varepsilon}{\|y-c\|}(y-c) \right\| \leq \|x_i - y\| + \varepsilon < \|x_i - x\|$$

It follows that  $z \in W(x)$ .

(ii) From (i) follows that  $W(x)$  is in a distance of at least  $\varepsilon$  from the most distant point of  $W^\varepsilon(x)$  so that  $W^\varepsilon(x) \subseteq D(c,d+2r-\varepsilon)$ . Combining this with Lemma 1 (ii) we have what was claimed, i.e.  $W^\varepsilon(x) \subseteq D(c,d+2r-\varepsilon) \setminus D(x,\varepsilon)$ .

\*\*\* Figure 2 about here \*\*\*

Figure 2 illustrates the “donut” shape of the boundary set that contains  $W^\varepsilon(x)$  for the example of a committee with three agents with ideal points in a two-dimensional space. Whilst the circle centered at  $x$  with radius  $d+2r-\varepsilon$  represents the upper bound of the  $\varepsilon$ -winset, no points in an  $\varepsilon$ -environment of  $x$  can be altered and are thus not part of the  $\varepsilon$ -winset. From Lemma 2 follows:

**PROPOSITION 1:** Let  $c$  be the center and  $r$  be the radius of the yolk. If  $\varepsilon > r$  then  $c \in C^\varepsilon$ .

*Proof.* Consider the case  $x = c$ . According to Lemma 2,  $W^\varepsilon(c) \subseteq D(c, 2r-\varepsilon) \setminus D(c, \varepsilon)$  so that  $W^\varepsilon(c) = \emptyset$  if  $\varepsilon > r$ .

In words, Proposition 1 states that if the costs associated with the act of voting are sufficiently larger than zero, the  $\varepsilon$ -core of the spatial voting game will be not empty. More specifically if  $\varepsilon$  is larger than the radius of the yolk, the yolk center will be an element of the  $\varepsilon$ -core. This is a noteworthy result as the yolk generally shrinks as the number of voters increases in large committees or in mass elections with a large number of voters (Koehler 1990; Tovey 1990; Hug 1999; Godfrey and Miller 2005). In this case, the center of the yolk is an element of the  $\varepsilon$ -core even for rather small voting thresholds. Moreover, for the special case of a random sample of ideal points drawn from a centered (e.g., uniform or normal) distribution, the  $\varepsilon$ -core exists with probability 1 for *any*  $\varepsilon > 0$  as the number of ideal points goes to infinity (Tovey 1991: 11). Consequently, for large electorates the center of

the yolk is a likely stable outcome. The following section shows, however, that a different point in the space is an even better approximation: the finagle point.

#### 4. MINIMAL VOTING THRESHOLD AND THE FINAGLE POINT

Given the stability result for sufficient large voting thresholds, a subsequent question is: what are the minimal thresholds that guarantee the existence of stable outcomes? This section shows that the minimal decision cost or threshold for the  $\varepsilon$ -core not to be empty is equal to the minimal finagle radius. In this case the finagle point is an element of the corresponding  $\varepsilon$ -core.

The finagle point is the point in the policy space with a minimal finagle radius: “from it a candidate can, with only minuscule changes in his initial policy location, find a response to any challenger that will defeat that challenger” (Wuffle et al. 1989: 348). The finagle point is close to the center of the yolk but in general does not coincide with it. More formally, the following definitions are used (Wuffle et al. 1989: 350f.):

**DEFINITION 2:** (i) A set of points  $V$  is a V-Mess for a majority voting game if for any  $x \notin V$  there is a  $y \in V$  so that  $y \in W(x)$ , i.e. it is externally stable.

(ii) The point that minimizes the distance to the maximally distant point in  $V$  is the center of  $V$ , the maximum distance is the radius of  $V$ .

(iii) The finagle radius of a point is the radius of the minimal circular V-Mess centered around it. The point with minimal finagle radius is the finagle point.

(iv) The minimal voting threshold  $\varepsilon^*$  is the minimal voting threshold  $\varepsilon \geq 0$  so that  $C^\varepsilon \neq \emptyset$ .

**PROPOSITION 2:** Let  $f$  be the finagle point,  $r_f$  the minimal finagle radius, and  $\varepsilon^*$  the minimal voting threshold for a majority voting game. Then:

$$\varepsilon^* = r_f \quad \text{and} \quad f \in C^{\varepsilon^*}.$$

*Proof.* (i) If the conventional core is empty,  $C \neq \emptyset$ , then  $\varepsilon^* = 0$ . But since the number of agents,  $n$ , is odd, any  $x \in C$  is a V-Mess so that  $r_f = 0$ . (ii) In general, however,  $C = \emptyset$ . In this case we have  $\varepsilon^* > 0$ , but there is also an upper bound for  $\varepsilon$  because for sufficiently large thresholds, there is always some alternative that is stable. I now show that for any circular V-Mess, its center is an element of the  $\varepsilon$ -core if  $\varepsilon$  equals its radius (or is larger). Formally, let  $D(x, r_x)$  be the open disk with center  $x$  and radius  $r_x$ , let  $\overline{D(x, r_x)}$  be its closure and let

$$V = \{ \overline{D(x, r_x)} : x \in \mathfrak{R}^k, r_x > 0, \overline{D(x, r_x)} \text{ is V-Mess} \}$$

denote the set of all circular V-Messes. We wish to show that

$$x \in C^\varepsilon \text{ for any } \overline{D(x, r_x)} \in V.$$

To see this, we have to show that for any  $\overline{D(x, r_x)}$ , the winset  $W^{r_x}(x)$  is empty, i.e. there is no alternative  $y \in W^{r_x}(x)$ . By way of contradiction, suppose that there is a  $y \in X$  with  $y \in W^{r_x}(x)$ . There are two cases:

Case 1:  $y \in \overline{D(x, r_x)}$ . Then  $\|y - x\| \leq r_x$  thus  $y \notin W^{r_x}(x)$ . Hence we have a contradiction.

Case 2:  $y \notin \overline{D(x, r_x)}$ . Then, as  $\overline{D(x, r_x)}$  is a V-Mess, there is at least one alternative that dominates  $y$ , i.e. there is a  $\chi \in X$  with

$$\|x - \chi\| \leq r_x, \quad (\text{a})$$

and a winning coalition  $S \in \mathcal{W}$  so that for any coalition agent's ideal point  $x_i$ :

$$\|x_i - \chi\| < \|x_i - y\|. \quad (\text{b}).$$

For the triangles  $x\chi x_i$  we have:

$$\left| \|x_i - \chi\| - \|x_i - x\| \right| < \|\chi - x\| \quad (\text{c})$$

Combining (a) to (c), we have  $\|x_i - y\| - \|x_i - x\| > \|x_i - \chi\| - \|x_i - x\| > -\|x - \chi\| \geq -r_x$ .

In words, this means that alternative  $x$  is weakly preferred to alternative  $y$  which implies that  $y$  cannot be element of the winset of  $x$ , i.e.  $y \notin W^x(x)$ . This, however, is a contradiction.<sup>4</sup>

Proposition 2 provides an alternative proof of Proposition 1 as the radius of the yolk is an upper bound for the minimal finagle radius (Wuffle et al. 1989: 358). For most configurations, however, the minimal finagle radius is much smaller than the yolk radius. Figure 3 illustrates Proposition 2 with respect to the three-voter case. Whilst the yolk is tangent to any side of the triangle, the minimal finagle radius is the radius of the inner circle ( $\varepsilon^*$ ) that is tangent to all three outer circles. The outer circles are centered at agents' ideal points and pass through those points on the triangle where the yolk is tangent. The minimal finagle radius represents the smallest decision costs or voting threshold where a majority-rule equilibrium exists. In this case, the finagle point ( $f$ ) rather than the yolk center ( $c$ ) is a stable outcome.

\*\*\* Figure 3 about here \*\*\*

For the three-actor case, the minimal finagle radius is always smaller than the yolk radius.

With three actors and ideal points  $x_1, x_2$  and  $x_3$ , the minimal finagle radius is given by:

$$r_f = \frac{1}{\alpha + \beta + \gamma + 2\sqrt{\alpha\beta + \beta\gamma + \alpha\gamma}}, \text{ where } \alpha = \frac{1}{p - \|x_2 - x_3\|}, \beta = \frac{1}{p - \|x_1 - x_3\|},$$

$$\gamma = \frac{1}{p - \|x_1 - x_2\|}, \text{ and } p = \frac{\|x_1 - x_2\| + \|x_2 - x_3\| + \|x_1 - x_3\|}{2}.$$

As a result, the ratio of the minimal finagle radius and that of the yolk is at most  $r_f / r = (2 + \sqrt{3})^{-1} \approx 0.267$  (Wuffle et al. 1989: 362). Is this also an upper bound for, at least generically, all ideal point configurations in the  $n$ -actor case? The following section shows that whilst this is not the case, the average ratio of minimal threshold to yolk radius in fact is considerably smaller than .267 and simulation results suggest that it is decreasing with the number of actors.

## 5. ESTIMATING THE MINIMAL THRESHOLD

Analytical methods would be preferable for estimating the threshold  $\varepsilon^*$  above which stable outcomes under simple majority rule exist for a specific configuration of ideal points. While the question can easily be answered for the 3-actor case, geometric or algebraic procedures are cumbersome even for 5-actor committees and become increasingly awkward for larger committees. Here simulation results can extend analytic conclusions as they esti-

mate empirical parameters rather than analytical bounds, in particular when we are interested in questions of the type “How likely are stable outcomes in typical committees” instead of investigating whether there exist rare, non-generic cases where the parameter in question is exceptionally large (or small).

To analyze the present setting, in which proposals are required to be at least a finite distance closer to a committee member’s ideal point to gain his vote, a computational model was built. To this end, first the number of ideal points  $n$  was selected randomly from a uniform distribution over the set of odd numbers in the range  $3 \leq n \leq 101$ . Second,  $n$  ideal points were selected from a uniform distribution over a two-dimensional “policy space”  $X = [0,10] \times [0,10]$ .<sup>5</sup> Given such a committee situation, an upper bound for the minimal epsilon that guarantees a stable outcome and the spatial location of these outcomes were estimated.<sup>6</sup> This procedure was carried out 4000 times generating 4000 ideal point configurations and estimates of the threshold  $\varepsilon^*$  and the stable outcome  $o^*$ . The results of this large number of experiments are reported in the subsequent tables and figures.

The algorithm used to estimate  $\varepsilon^*$  and  $o^*$  calculates an approximation  $C(\varepsilon)$  of the  $\varepsilon$ -core for a sufficient large  $\varepsilon$  and then decreases  $\varepsilon$  gradually until the area of  $C(\varepsilon)$  is reasonably small so that  $\varepsilon$  is an approximation for  $\varepsilon^*$ .<sup>7</sup> The algorithm proceeds in three steps:

Step 1: Use two times the yolk radius as the initial value for the voting threshold ( $\varepsilon = 2r$ ) and the yolk center as the initial value for the center of the  $\varepsilon$ -core ( $z = c$ ). We know from Proposition 2 that for  $\varepsilon = r$ ,  $c \in C(\varepsilon)$ . As we do not know whether or not  $r$  is the least upper bound of  $\varepsilon^*$  (so that the interior of  $C(r)$  is empty) we use two times the yolk radius to ensure that the initial  $C(r)$  has a non-empty interior.

Step 2: Approximate location and size of  $C(\varepsilon)$  by finding the boundary points of  $C(\varepsilon)$  on any ray from  $z$  specified by angle  $\theta = 2\pi k/16$  for  $k = 1, \dots, 16$ . The resulting  $16n$ -polygon is a subset of  $C(\varepsilon)$  with area  $A$ , and center of gravity  $c_P$ . If  $A < \pi r^2/10^{10}$ , set  $\varepsilon^* = \varepsilon$  and  $o^* = c_P$ .<sup>8</sup> Otherwise proceed with step 3.

Step 3: Denote by  $r_P$  the radius of the largest circle with center  $c_P$  that can be inscribed in the  $16n$ -polygon. The new threshold  $\varepsilon$  then is  $\varepsilon - r_P/2^m$  (where  $m$  is chosen so that  $W(c_P)$  is empty), the new approximate center of the  $\varepsilon$ -core is  $z = c_P$ . Proceed with step 2.

The upper part of Table 1 shows the distribution of committee sizes and members' ideal points for the 4000 experiments that were run. The figures do not deviate from what we would expect to see. The two-dimensional mean of committee members' ideal points is, on average, close to the center of the space at (5;5) with an average two-dimensional variance of 16.6 as expected for a uniform distribution of ideal points.<sup>9</sup> Results on the spatial location of the yolk and the  $\varepsilon^*$ -core are shown in the lower part of Table 1. Yolk centers and approximate  $\varepsilon^*$ -cores cluster in the middle of the policy space as expected (with means of about 4.99 and standard deviations of .76 on each of the two dimensions). Whilst the average yolk radius is considerable (.5807), the simulation results reveal that the minimal threshold that guarantees the existence of a stable outcomes is on average much smaller, ranging from .0002 to .63 with a mean threshold size of .0755. More specifically, the  $\varepsilon^*$ -to-yolk radius ratio is small with an average ratio of .1159 and a maximum ratio of .3523. This is in line with Proposition 1 stating that the  $\varepsilon$ -core is non-empty for  $\varepsilon > r$  but it also sug-

gests that the smallest upper bound of the ratio for the 3-actor case which is  $r_f / r \approx 0.267$  is not an upper bound for the  $n$ -actor case.

\*\*\* Table 1 about here \*\*\*

Figure 4 shows the location of the yolk center and the  $\varepsilon^*$ -core for the first 100 experiments that were run. Outcomes are located almost anywhere in the two-dimensional alternative space but the yolk center and  $\varepsilon^*$ -core are always located in a small distance to each other. Yet, they never coincide.<sup>10</sup> Moreover, Table 1 shows that the distance is always smaller than the yolk radius suggesting that the  $\varepsilon^*$ -core or finagle point is an element of the yolk (cp. Wuffle et al. 1989: 362f.).

\*\*\* Figure 4 about here \*\*\*

Figures 5-7 present three panels reporting the yolk radius, the estimated minimal voting threshold and the minimal threshold-to-yolk radius ratio by committee size. The first panel replicates findings from previous analyses of the yolk showing that the yolk radius on average decreases as the size of the committee increases. The same is true for the variance in the yolk radius for a given committee size. This is consistent with previous simulation results, e.g. those reported in Koehler (1990) and Hug (1999). The second panel shows the magnitude of the minimal threshold by committee size. For a three actor committee, decision costs of an average of no more than .2 are sufficient to guarantee stable outcomes un-

der simple majority rule. With 23 agents, the average minimal threshold decreases to .1, with a 101-agent committee the average threshold is .03. The third panel presents the key finding showing the minimal threshold-to-yolk radius ratio by committee size. Three findings are noteworthy: first, “confirming” the analytical findings, the minimal threshold for a 3-agent committee is at most .2679. Second, both the average ratio and the variance in the ratio decreases with the committee size. Third, at least for small committee sizes ( $3 < n \leq 15$ ) there are ideal point configurations where  $\varepsilon^*$  exceeds the 3-actor-boundary of .2679. Moreover, despite the fact that the ratio is decreasing for large  $n$ , the average  $\varepsilon^*$  for the 5-actor committee is larger than that for the 3-actor committee. Finally, replicating the analysis using a bivariate normal distribution to draw actors’ ideal points, yields similar results. In a sample of 4000 experiments, the  $\varepsilon^*$ -to-yolk radius ratio has an average of .1164, the mean ratio between the yolk center-to- $\varepsilon^*$ -core distance and the yolk radius is .2344.

\*\*\* Figures 5-7 about here \*\*\*

This all is consistent with Koehler’s (2001: 166) simulation results. In his setting, stable outcomes are generated when, on average,  $\varepsilon/r = 0.279$  and  $n = 72.733$ , while simulations were run for odd-sized committees ( $21 \leq n \leq 101$ ) with ideal points selected randomly from a uniform distribution over a two-dimensional square. Supplementing these findings, the present study shows that the actual average minimal threshold  $\varepsilon^*$  is much smaller.

## 6. CONCLUSION

What does that all mean for the analysis of committee voting? Essentially, there are two scenarios. First, if the distribution of voters' ideal points is somehow symmetric, then a median in all directions or a stable outcome will exist in the conventional spatial model. In this case,  $\varepsilon^*$  equals zero and the median in all directions, the yolk center, the finagle point and the  $\varepsilon$ -core coincide. Secondly and from a generic standpoint, if the voter distribution deviates from symmetry, a multi-dimensional median will not exist, and the yolk center will generally be different from the finagle point. In this case, and if the act of voting is not costless for the committee members but comes with some decision costs, the finagle point rather than the center of the yolk represents the best approximation of likely outcomes. The finding is in line with theoretical, experimental and simulation analyses which show that stable outcomes under restricted voting can be found "near the center of the preference space" (Koehler 2001: 166). The above analysis suggests that this center of the preference space is the finagle point rather than the yolk center. Finally, the minimal voting threshold that guarantees stable outcomes is, even for modest committee sizes, small and decreases as the committee size increases. This suggests that given the strict and probably unreasonable assumptions of the conventional spatial model as regards the costlessness of voting, stable, preference- and not only institution-induced outcomes can indeed be expected in real world politics.

## NOTES

<sup>1</sup> Note that this says nothing about the “desirability” of stability. In settings where undominated alternatives abound, e.g. under unanimity rule, rival coalitions have unlike preferences as to where to move a distant status quo to within the set of undominated outcomes. As further changes will be impossible this might cause deadlock. This is the very reason why in comparative politics, policy stability is often considered to be negatively related to regime and/or government stability (e.g., Tsebelis 2002: 2-6).

<sup>2</sup> It might be argued that devising a proposal incrementally better for a majority of voters is much more costly for the agenda-setter than figuring out for a particular committee member what he or she prefers. This would result in model with an (additional) restriction on the preference maximizing of the agenda-setter. In this paper, I refrain from considering these proposal costs. For a different setup where  $\varepsilon$  is an agent-specific decision cost and  $x$  has to be at least a certain *percentage* closer to the individuals ideal points than  $y$ , e.g.  $x$  is preferred to  $y$  if and only if  $\|x_i - x\| < (1 - \varepsilon_i) \|x_i - y\|$ , see Sloss (1979), and Ganghof and Bräuninger (2006).

<sup>3</sup> More specific bounds can be defined in terms of cardioids with center  $c$ , cusps at  $x$ , radii  $d$  and eccentricities of  $2r$  and  $-2r$ .

<sup>4</sup> Note that if  $C^\varepsilon$  is strictly convex for any  $\varepsilon > 0$ , then one can show that the  $\varepsilon$ -core coincides with the finagle point, i.e.  $\{f\} = C^{\varepsilon^*}$ . Otherwise one could always find an open ball with center  $z$  in the interior of the  $\varepsilon$ -core and a  $\delta$  such that  $z \in (\varepsilon - \delta)$ -core. Convexity of  $C^\varepsilon$ ,

however, is not straightforward as the Pareto set of an arbitrary winning coalition generally is not convex.

<sup>5</sup> The simulation is limited to two dimensions for three reasons. First, most analyses of spatial models of committee voting are limited to two dimensions; thus sticking to the two-dimensional case will allow for direct comparison to previous studies (Wuffle et al. 1989; Hug 1999; Koehler 2001). Second, two-dimensional results usually apply to higher dimensions (Miller et al. 1989; Koehler 1996). Third, a huge body of studies of a variety of political systems has demonstrated persuasively that most legislative decisions can be represented and explained on two dimensions (e.g. Poole and Rosenthal 2000). Finally, the choice of a uniform rather than normal random distribution of ideal points is also for the sake of comparability with the study of Koehler (2001). I outline results for a bivariate normal distribution in the text.

<sup>6</sup> Note that if  $C^\varepsilon$  is strictly convex,  $C^{\varepsilon^*}$  will contain exactly one point so that the estimated upper bound for  $\varepsilon^*$  is also an approximation for  $\varepsilon^*$ .

<sup>7</sup> Yolks and  $\varepsilon^*$ -cores are calculated using the program *Mineps* (available at <http://www.tbraeuninger.de/download.html>) which requires the commercial program *Gauss*.

<sup>8</sup> This means that the algorithm stops if the approximate area of the  $\varepsilon$ -core is  $10^{10}$  times smaller than the area of the yolk.

<sup>9</sup> For a uniform distribution over the 10-by-10 square the expected two-dimensional variance is  $V = 10^2(n - 1) / 6n$ , where  $n$  is the number of ideal points (Koehler 1996: 161). Thus

for committee sizes  $n = 3, \dots, 101$  drawn from a uniform distribution the expected variance

$$\text{is } \sum_{n=3}^{101} \frac{100 \cdot 2n}{6(2n+1)} = 16.02 .$$

<sup>10</sup> Yolk center and  $\varepsilon^*$ -core coincide, for instance, when the ideal points of three agents form an equilateral triangle, and generally, for any symmetric ideal point configuration.

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Table 1: Summary statistics of 4000 experiments

	Mean	Std. Dev.	Min	Max
Number of actors	52.0075	28.8449	3	101
Mean of ideal point position				
$x$ coordinate	5.0017	.5663	1.2613	9.6488
$y$ coordinate	4.9876	.5852	.9688	9.4697
Standard deviation of ideal point position				
$x$ coordinate	2.8705	.3099	.1498	4.8152
$y$ coordinate	2.8636	.3230	.2173	5.1182
Two-dimensional variance	16.6404	2.4546	1.6576	40.6692
Yolk radius ( $r$ )	.5807	.2482	.0076	2.3956
Yolk center				
$x$ coordinate	4.9958	.7501	.9696	9.6776
$y$ coordinate	4.9815	.7554	.8317	9.5769
Estimate of minimal threshold ( $\varepsilon^*$ )	.0755	.0671	.0002	.6337
Approximate location of $\varepsilon^*$ -core				
$x$ coordinate	4.9948	.7680	.8913	9.7816
$y$ coordinate	4.9813	.7694	.5956	9.4043
Distance yolk center-to- $\varepsilon^*$ -core	.1193	.0815	.0009	.6141
Ratio of distance yolk center-to- $\varepsilon^*$ -core to yolk radius	.2162	.1389	.0015	0.9630
Ratio of $\varepsilon^*$ to yolk radius	.1159	.0471	.02356	.3523

$N = 4000$  experiments; number of actors drawn from a uniform distribution over the set of odd numbers between 3 and 101; ideal points drawn from a uniform distribution over the square  $[0,10] \times [0,10]$ .

Figure 1: Preferred sets and winsets for a three-member committee

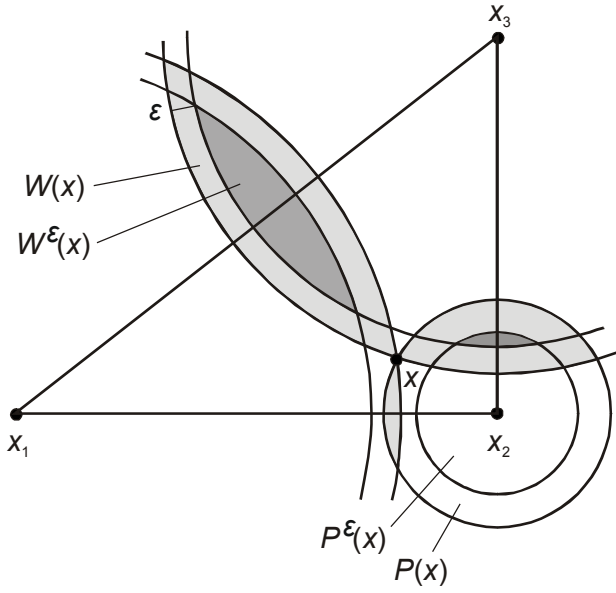


Figure 2: Donut shape of boundary of  $\varepsilon$ -winset for a three-member committee

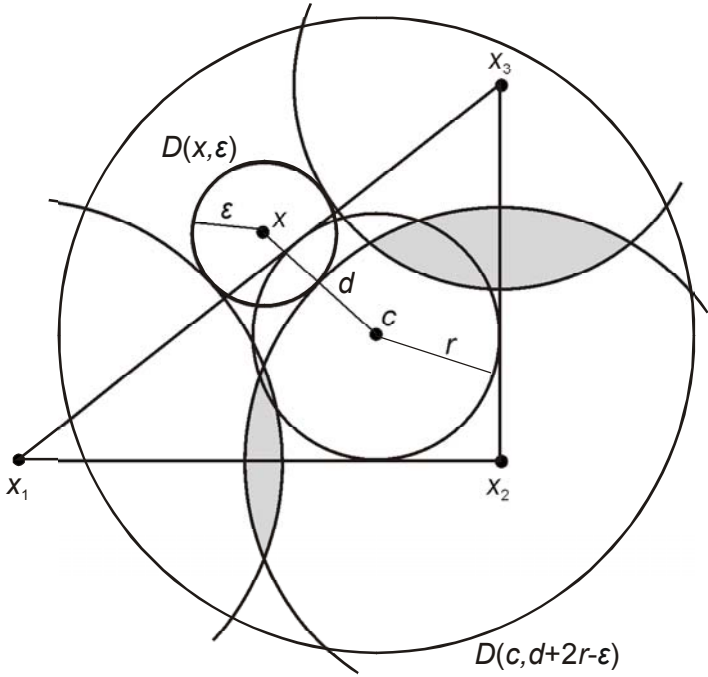


Figure 3: Minimal threshold  $\varepsilon^*$  for a three-member committee

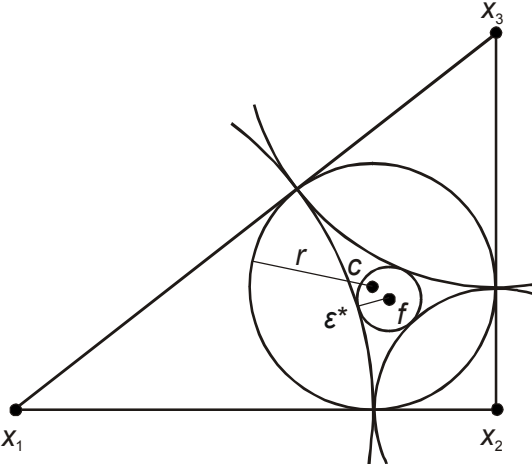


Figure 4: Pairing up of yolk centers (+) and  $\epsilon^*$ -cores (o) for 100 experiments

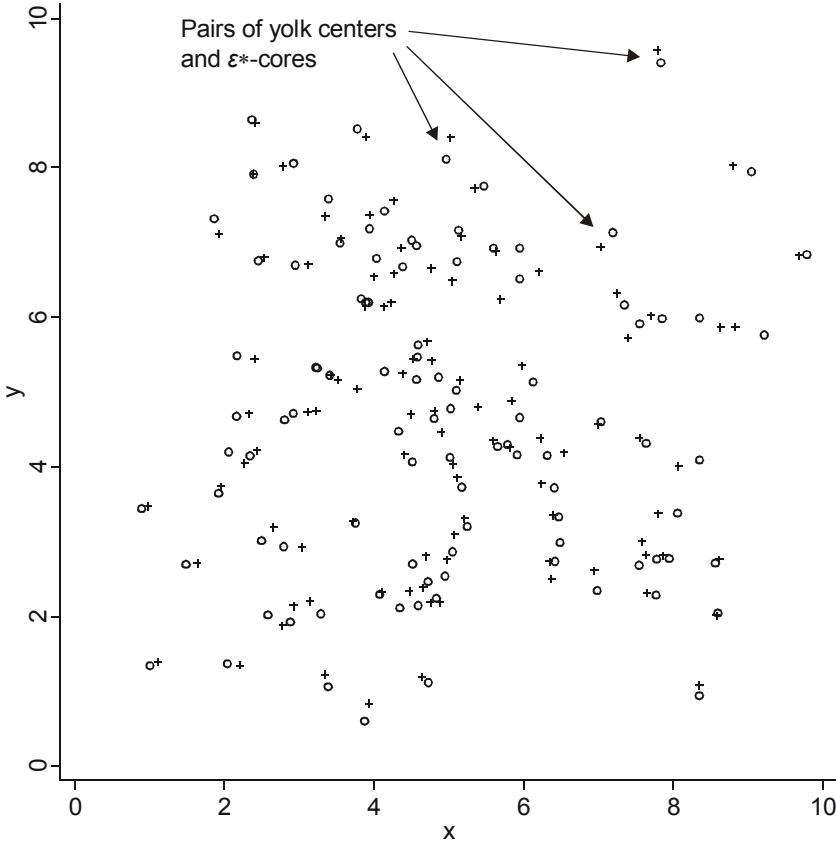
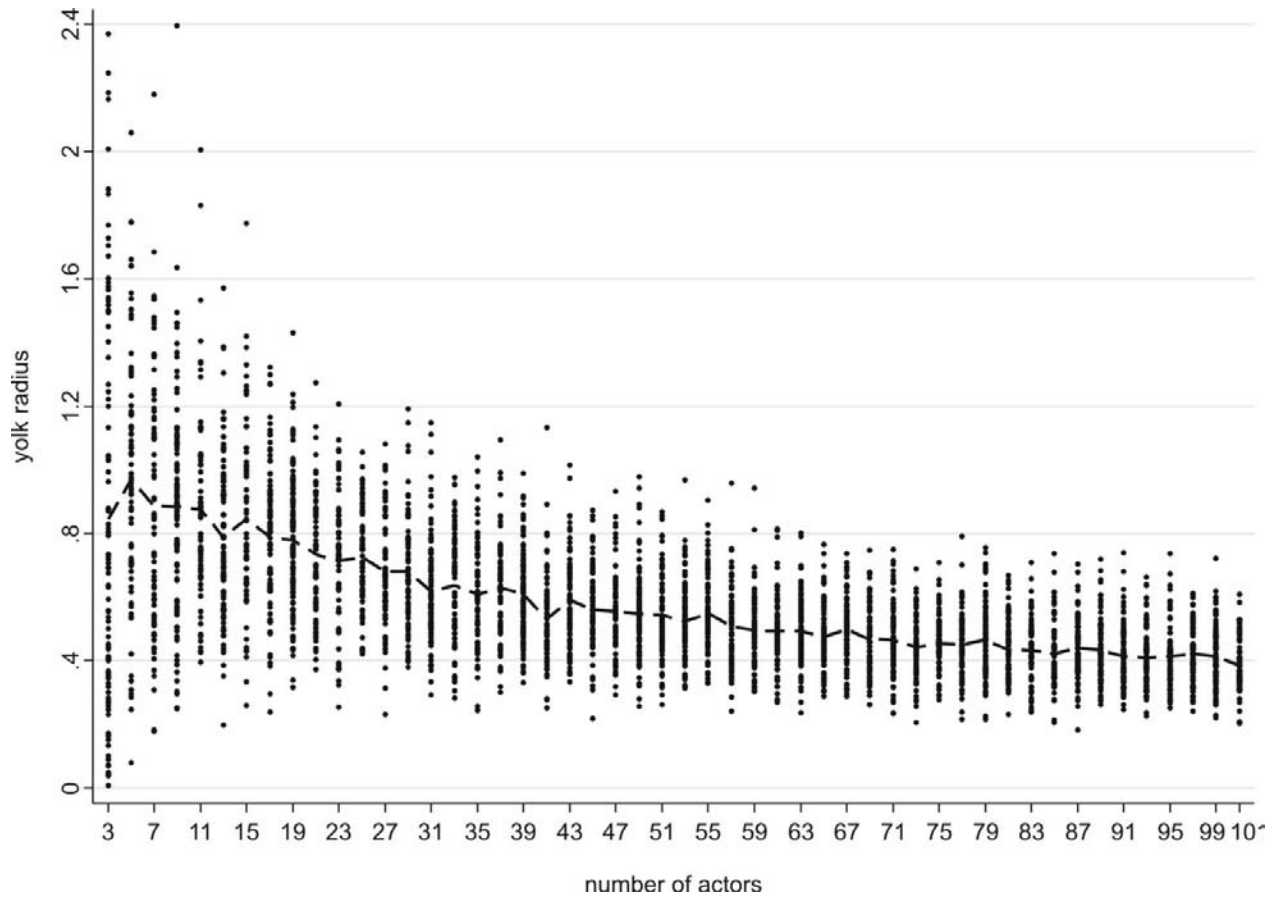
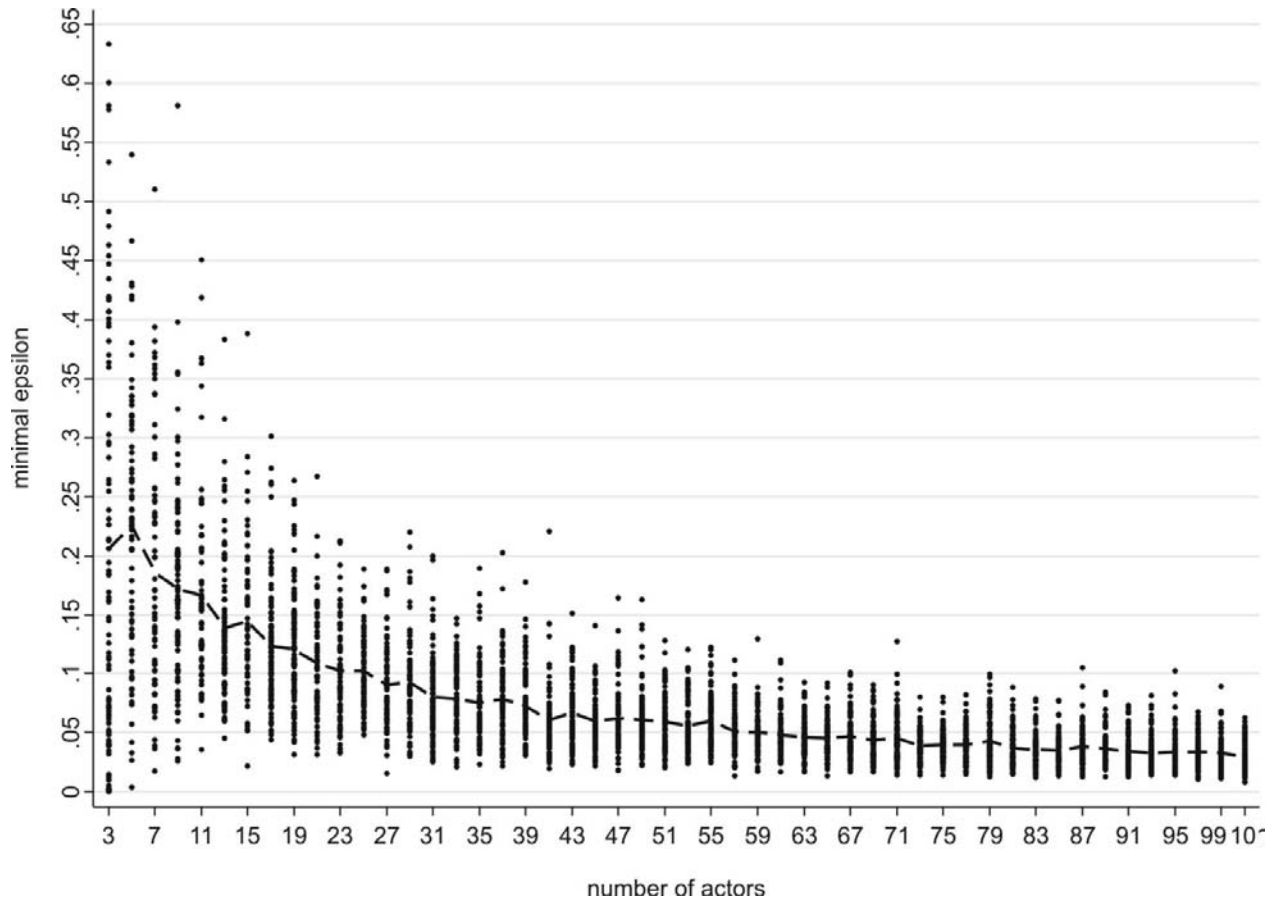


Figure 5: Yolk radius by committee size



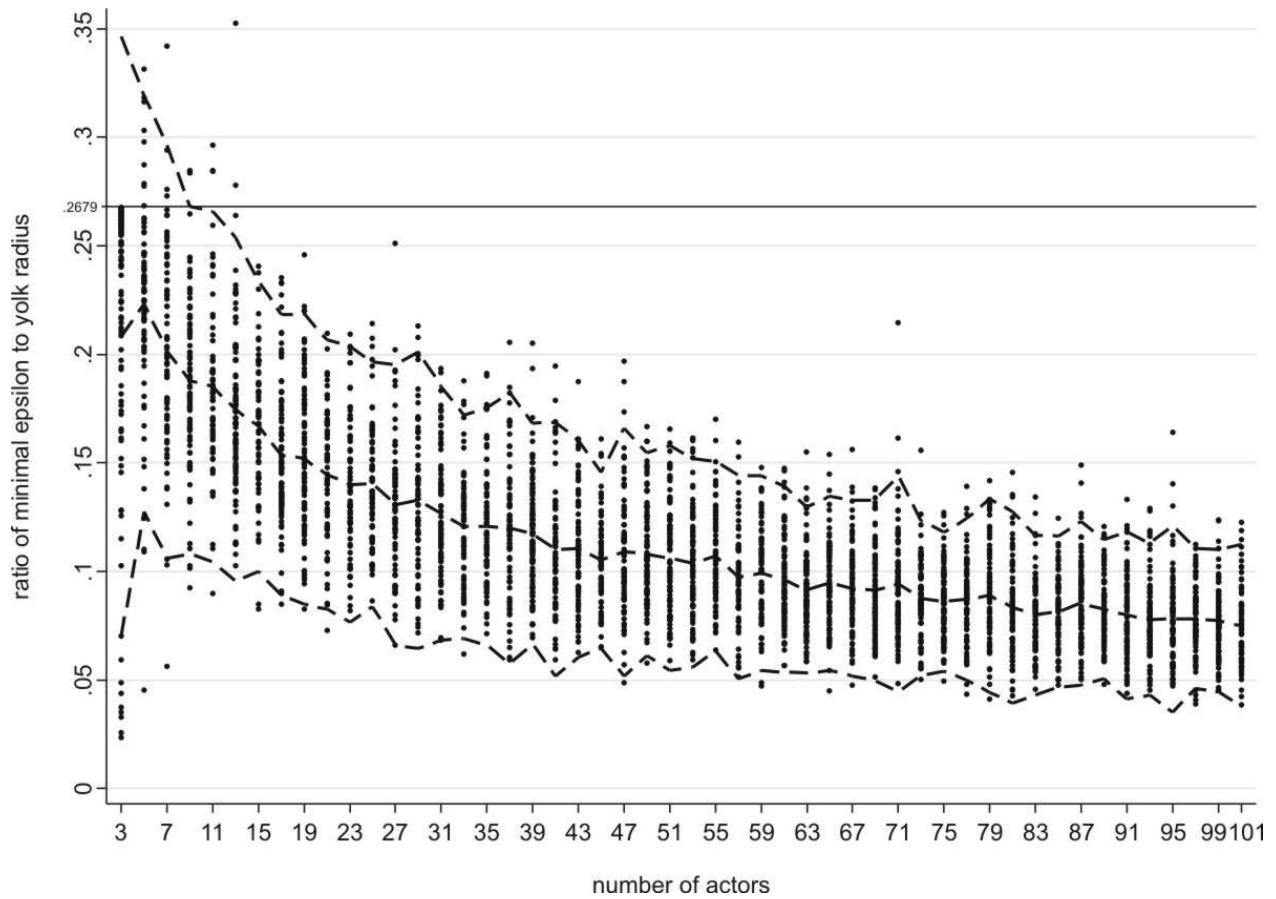
Note: The solid line is the average yolk radius.

Figure 6: Minimal threshold by committee size



Note: The solid line is the average minimal threshold.

Figure 7: Ratio of minimal threshold to yolk radius by committee size



Note: The solid line is the average ratio, dashed lines are upper and lower bound ratios (mean  $\pm$  2 standard deviations).